

Angular momentum conservation law in Einstein-Cartan space-time

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In the light of the local Lorentz transformations and the general Noether theorem, a new formulate of the general covariant angular momentum conservation law in Einstein-Cartan gravitation theory is obtained, which overcomes the critical difficulty in the other formulates that the conservation law depended on the coordinative choice.

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I. INTRODUCTION

Conservation law of energy-momentum and angular momentum have been of fundamental interest in gravitational physics [1]. Using the vierbein representation of general relativity, Duan *et al* obtained a general covariant conservation law of energy-momentum which overcomes the difficulties of other expressions [2]. This conservation law gives the correct quadrupole radiation formula of energy which is in good agreement with the analysis of the gravitational damping for the pulsar PSR1916-13 [3]. Also, from the same point of view, Duan and Feng [4] proposed a general covariant conservation law of angular momentum in Riemannian space-time which does not suffer from the flaws of the others [5–7].

On the other hand, though the Einstein theory of general relativity and gravitation has succeeded in many respects, there is an essential difficulty in this theory: we could not get a successful renormalized quantum gravity theory [8]. In order to find renormalized theory, many physicists [9,10] have studied this problem in its more general aspects, i.e. extending Einstein's theory to Einstein-Cartan theory, which includes torsion tensor [11].

As is well known, torsion is a slight modification of the Einstein's theory of relativity [12], but is a generalization that appears to be necessary when one tries to conciliate general relativity with quantum theory. Like opening a Pandora's box, many works have been done in this region [13,14]. Today, general relativity with non-zero torsion is a major contender for a realistic generalization of the theory of gravitation.

About two decades ago, Helh [11] gave, in Einstein-Cartan theory, an expression of the angular momentum conservation law which was worked out from Noether's theorem, but in that expression, all quantities carried Riemannian indices and the total angular momentum depended on the coordinative choice which is not an observable quantity. Some physicists [9,15] investigated the same problem from different viewpoints and presented other expressions of conservation law which is not general covariant, hence this theory cannot be said to be very satisfactory.

Several years ago, the general covariant energy-momentum conservation law in general space-time has been discussed successfully by Duan *et al* [16]. In this paper, we will discuss the angular momentum conservation law in Einstein-Cartan theory via the vierbein representation. General Relativity without vierbein is like a boat without a jib—without these vital ingredients the going is slow and progress inhibited. Consequently, vierbein have grown to be an indispensable tool in many aspects of general relativity. More important, it is relevant to the physical observability. Based on the Einstein's observable time and space interval, we take the local point of view that any measurement in physics is performed in the local flat reference system whose existence is guaranteed by the equivalence principle, i.e. an observable object must carries, instead of the indices of the space-time coordinates, the indices of internal space. Thus, we draw the support from vierbein not only for mathematical reason, but also because of physical measurement consideration.

II. CONSERVATION LAW IN GENERAL CASE

The conservation law is one of the important essential problems in gravitational theory. It is due to the invariant of Lagrangian corresponding to some transformation. In order to study the general covariant angular momentum

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conservation law, it is necessary to discuss conservation law by the Noether theorem in general case.

The action of a system is

$$I = \int_M \mathcal{L}(\phi^A, \phi_{,\mu}^A) d^4x, \quad (1)$$

where ϕ^A are independent variable with general index A , $\phi_{,\mu}^A = \partial_\mu \phi^A$. If the action is invariant under the infinitesimal transformation

$$x'^\mu = x^\mu + \delta x^\mu, \quad (2)$$

$$\phi'^A(x') = \phi^A(x) + \delta \phi^A(x), \quad (3)$$

and $\delta \phi^A$ is zero on the boundary of the four-dimensional volume M , then we can prove that there is the relation

$$\frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}^A}) + [\mathcal{L}]_{\phi^A} \delta_0 \phi^A = 0 \quad (4)$$

where $[\mathcal{L}]_{\phi^A}$ is

$$[\mathcal{L}]_{\phi^A} = \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}^A}), \quad (5)$$

and $\delta_0 \phi^A$ is the Lie derivative of ϕ^A

$$\delta_0 \phi^A = \delta \phi^A(x) - \phi_{,\mu}^A \delta x^\mu. \quad (6)$$

If \mathcal{L} is the total Lagrangian density of the system, there is $[\mathcal{L}]_{\phi^A} = 0$, the field equation of ϕ^A with respect to $\delta I = 0$. From the above equation, we know that there is a conservation equation corresponding to the above transformations

$$\frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}^A} \delta_0 \phi^A) = 0 \quad (7)$$

This is just the conservation law in general case. It must be pointed out that if \mathcal{L} is not the total Lagrangian density of the system, then as long as the action of \mathcal{L} remains invariant under these transformations, (4) is still tenable. But (7) is not admissible now due to $[\mathcal{L}]_{\phi^A} \neq 0$.

In gravitation theory with the vierbein as element fields, we can separate ϕ^A as $\phi^A = (e_\mu^a, \psi^B)$, where e_μ^a is the vierbein field and ψ^B is an arbitrary tensor under general coordinate transformation. When ψ^B is $\psi^{\mu_1 \mu_2 \dots \mu_k}$, we can always scalarize it by

$$\psi^{a_1 a_2 \dots a_k} = e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} \dots e_{\mu_k}^{a_k} \psi^{\mu_1 \mu_2 \dots \mu_k},$$

so we can take ψ^B as a scalar field under general coordinate transformations. In later discussion we can simplify the equations by such a choice.

III. GENERAL COVARIANT CONSERVATION LAW OF ANGULAR MOMENTUM IN EINSTEIN-CARTAN THEORY

As is well known, in Einstein-Cartan theory, the total action of the gravity-matter system is expressed by

$$I = \int_M \mathcal{L} d^4x = \int_M (\mathcal{L}_g + \mathcal{L}_m) d^4x, \quad (8)$$

$$\mathcal{L}_g = \frac{c^4}{16\pi G} \sqrt{-g} R \quad (9)$$

\mathcal{L}_g is the gravitational Lagrangian density, R is the scalar curvature of the Riemann-Cartan space-time. The matter part Lagrangian density \mathcal{L}_m take the form $\mathcal{L}_m = \mathcal{L}_m(\phi^A, D_\mu \phi^A)$, where the matter field ϕ^A belongs to some representation of Lorentz group whose generators are I_{ab} ($a, b = 0, 1, 2, 3$) and $I_{ab} = -I_{ba}$, D_μ is the covariant derivative operator of ϕ^A

$$D_\mu \phi^A = \partial_\mu \phi^A - \frac{1}{2} \omega_{\mu ab} (I_{ab})^A_B \phi^B. \quad (10)$$

where $\omega_{\mu ab}$ is the spin connection.

As in Einstein-Cartan theory, the affine connection $\Gamma_{\mu\nu}^\lambda$ is not symmetry in μ and ν , i.e. there exists non-zero torsion tensor

$$T_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda. \quad (11)$$

It is well known, for vierbein field e_μ^a , the total covariant derivative is equal to zero, i.e.

$$\mathcal{D}_\mu e_\nu^a \equiv \partial_\mu e_\nu^a - \omega_{\mu ab} e_\nu^b - \Gamma_{\mu\nu}^\lambda e_\lambda^a = 0. \quad (12)$$

From this formula, we can obtain another expression of torsion tensor

$$T_{\mu\nu}^a = T_{\mu\nu}^\lambda e_\lambda^a = D_\mu e_\nu^a - D_\nu e_\mu^a, \quad (13)$$

where $D_\mu e_\nu^a = \partial_\mu e_\nu^a - \omega_{\mu}^{ab} e_\nu^b$. In fact, this formula is just the Cartan structure equation.

The scalar curvature of the Riemann-Cartan space-time is expressed by

$$\begin{aligned} R = & e_a^\mu e_b^\nu \partial_\mu \omega_{\nu ab} - e_a^\mu e_b^\nu \partial_\nu \omega_{\mu ab} + \bar{\omega}_{bac} \bar{\omega}_{acb} + \bar{\omega}_b \bar{\omega}_b \\ & + \bar{\omega}_{acb} T_{acb} - 2\bar{\omega}_b T_b + T_b T_b + \frac{1}{2} T_{cba} T_{bac} + \frac{1}{4} T_{acb} T_{abc}. \end{aligned} \quad (14)$$

where $\omega_{abc} = e_a^\mu \omega_{\mu bc}$, $\omega_a = \omega_{bab}$ are the representation of spin connection, $\bar{\omega}_a = \bar{\omega}_{bab} = e_b^\mu (\partial_\mu e_a^\nu + \{\nu_{\mu\sigma}\} e_a^\sigma) e_{\nu b}$ in which $\{\nu_{\mu\sigma}\}$ is the Christoffel symbol. By tedious calculation, the above formulation allows us to obtain the identity

$$\mathcal{L}_g = \frac{c^4}{16\pi G} \sqrt{-g} R = \frac{c^4}{8\pi G} \mathcal{L}_\partial + \mathcal{L}_\omega \quad (15)$$

$$\mathcal{L}_\partial = \partial_\mu (\sqrt{-g} e_a^\mu \omega_a) \quad (16)$$

$$\mathcal{L}_\omega = \frac{c^4}{16\pi G} \sqrt{-g} [\omega_{bac} \omega_{acb} + \omega_a \omega_a - 2\bar{\omega}_a \omega_a - 2e_a^\mu \partial_\mu (e_b^\nu) e_{\nu c} \omega_{cab}] \quad (17)$$

It is well known that in deriving the general covariant conservation law of energy momentum in general relativity, the general displacement transformation, which is a generalization of the displacement transformation in the Minkowski space-time, was used [16]. In the local Lorentz reference frame, the general displacement transformation takes the same form as that in the Minkowski space-time. This implies that general covariant conservation laws are corresponding to the invariance of the action under local transformations. We may conjecture that since the conservation law for angular momentum in special relativity corresponds to the invariance of the action under the Lorentz transformation, the general covariant conservation law of angular momentum in general relativity may be obtained by means of the local Lorentz invariance.

we choose vierbein e_μ^a , spin connection ω_{abc} and the matter field ϕ^A as independent variables. Under the local Lorentz transformation

$$e_\mu^a(x) \rightarrow e_\mu'^a(x) = \Lambda_b^a(x) e_\mu^b(x), \quad \Lambda_c^a(x) \Lambda_b^c(x) = \delta_b^a, \quad (18)$$

ω_{abc} and ϕ^A transform as

$$\omega_{abc}(x) \rightarrow \omega_{abc}'(x) = \omega_{lmn}(x) \Lambda_a^l \Lambda_b^m \Lambda_c^n + \Lambda_a^d e_d^\mu \Lambda_b^l \partial_\mu \Lambda_c^l \quad (19)$$

$$\phi^A \rightarrow \phi'^A = D(\Lambda(x))^A_B \phi^B(x). \quad (20)$$

Since the coordinates x^μ do not transform under the local Lorentz transformation, $\delta x^\mu = 0$, from (6), it can be proved that in this case, $\delta_0 \rightarrow \delta$. It is required that \mathcal{L}_m is invariant under (18) and \mathcal{L}_g is invariant obviously. So under the local Lorentz transformation (18) \mathcal{L} is invariant. In the light of the discussion in section 2, we would like to have the relation

$$\begin{aligned} & \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu e_a^\nu} \delta e_a^\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A} \delta \phi^A + \frac{\partial \mathcal{L}}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} \right) \\ & + [\mathcal{L}]_{e_a^\nu} \delta e_a^\nu + [\mathcal{L}]_{\omega_{abc}} \delta \omega_{abc} + [\mathcal{L}]_{\phi^A} \delta \phi^A = 0 \end{aligned} \quad (21)$$

where $[\mathcal{L}]_{e_a^\nu}$, $[\mathcal{L}]_{\omega_{abc}}$ and $[\mathcal{L}]_{\phi^A}$ are the Euler expressions defined as

$$\begin{aligned} [\mathcal{L}]_{e_a^\nu} &= \frac{\partial \mathcal{L}}{\partial e_a^\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu e_a^\nu}, \\ [\mathcal{L}]_{\omega_{abc}} &= \frac{\partial \mathcal{L}}{\partial \omega_{abc}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \omega_{abc}}, \\ [\mathcal{L}]_{\phi^A} &= \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A}. \end{aligned}$$

Using the equations $[\mathcal{L}]_{e_a^\nu} = 0$, $[\mathcal{L}]_{\omega_{abc}} = 0$ and $[\mathcal{L}]_{\phi^A} = 0$, we get the following expression

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}_g}{\partial \partial_\mu e_a^\nu} \delta e_a^\nu + \frac{\partial \mathcal{L}_g}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} \right) + \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}_m}{\partial \partial_\mu e_a^\nu} \delta e_a^\nu + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A} \delta \phi^A \right) = 0 \quad (22)$$

where we have used the fact that only \mathcal{L}_m contain the matter field ϕ^A . Consider the infinitesimal local transformation $\Lambda_b^a(x) = \delta_b^a + \alpha_b^a(x)$, $\alpha_{ab} = -\alpha_{ba}$, $D(\Lambda)$ can be linearized as $[D(\Lambda)]_B^A = \delta_B^A + \frac{1}{2}(I_{ab})_B^A \alpha_{ab}$, we have

$$\delta e_a^\nu = \alpha_{ab} e_b^\nu(x), \quad (23)$$

$$\delta \omega_{abc}(x) = \alpha_{ad} \omega_{dbc} + \alpha_{bd} \omega_{adc} + \alpha_{cd} \omega_{abd} + e_a^\mu \partial_\mu (\alpha_{bc}) \quad (24)$$

$$\delta \phi^A = \frac{1}{2} (I_{ab})_B^A \phi^B(x) \alpha_{ab}(x) \quad (25)$$

We introduce j_{ab}^μ

$$\begin{aligned} \sqrt{-g} j_{ab}^\mu \alpha_{ab} &= \frac{3}{c} \left[\frac{\partial \mathcal{L}_\omega}{\partial \partial_\mu e_a^\nu} e_b^\nu \alpha_{ab} - \frac{\partial \mathcal{L}_m}{\partial \partial_\mu e_a^\nu} e_b^\nu \alpha_{ab} \right. \\ &\quad \left. - \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \omega_{abc}} (\alpha_{ad} \omega_{dbc} + \alpha_{bd} \omega_{adc} + \alpha_{cd} \omega_{abd} + e_a^\mu \partial_\mu (\alpha_{bc})) - \frac{1}{2} \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A} (I_{ab})_B^A \phi^B \alpha_{ab} \right], \end{aligned} \quad (26)$$

then (22) can be rewritten as

$$\partial_\mu (\sqrt{-g} j_{ab}^\mu \alpha_{ab}) + \frac{3c^3}{8\pi G} \partial_\mu \left(\frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu e_a^\nu} e_b^\nu \alpha_{ab} + \frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} \right) = 0. \quad (27)$$

From (16) one can easily get that

$$\frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu e_a^\nu} e_b^\nu \alpha_{ab} = \sqrt{-g} \omega_a e_b^\mu \alpha_{ab}, \quad (28)$$

$$\frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} = \sqrt{-g} e_b^\mu (\omega_a \alpha_{ba} + e_c^\nu \partial_\nu \alpha_{bc}), \quad (29)$$

considering that α_{ab} is antisymmetry, i.e. $\alpha_{ab} = -\alpha_{ba}$, we then get the result that

$$\frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu e_a^\nu} e_b^\nu \alpha_{ab} + \frac{\partial \mathcal{L}_\partial}{\partial \partial_\mu \omega_{abc}} \delta \omega_{abc} = \sqrt{-g} e_a^\mu e_b^\nu \partial_\nu \alpha_{ab}. \quad (30)$$

Defining a superpotential $V_{ab}^{\mu\nu} = e_a^\mu e_b^\nu - e_b^\mu e_a^\nu$, and substituting (30) into (27), we obtain

$$\partial_\mu (\sqrt{-g} j_{ab}^\mu) \alpha_{ab} + [\sqrt{-g} j_{ab}^\mu - (\frac{3c^3}{16\pi G}) \partial_\nu (\sqrt{-g} V_{ab}^{\nu\mu})] \partial_\mu \alpha_{ab} = 0. \quad (31)$$

Since α_{ab} and $\partial_\mu \alpha_{ab}$ are independent of each other, we must have

$$\partial_\mu (\sqrt{-g} j_{ab}^\mu) = 0, \quad (32)$$

$$j_{ab}^\mu = \frac{3c^3}{16\pi G} \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} V_{ab}^{\nu\mu}). \quad (33)$$

From (32) and (33), it can be concluded that j_{ab}^μ is conserved identically. Since the current j_{ab}^μ is derived from the local Lorentz invariance of the total Lagrangian, it can be interpreted as the total angular momentum tensor density of the gravity-matter system.

For a globally hyperbolic Riemann-Cartan manifold, there exist Cauchy surfaces Σ_t foliating M . We choose a submanifold D of M joining any two Cauchy surfaces Σ_{t_1} and Σ_{t_2} , so the boundary ∂D of D consists of three parts: Σ_{t_1} , Σ_{t_2} and A which is at spatial infinity. For an isolated system, the space-time should be asymptotically flat at spatial infinity, so the vierbein have the following asymptotical behaviour [2,4,17]

$$\lim_{r \rightarrow \infty} (\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}) = 0. \quad (34)$$

Since

$$\sqrt{-g} V_{ab}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} e_{\lambda c} e_{\rho d},$$

we have $\lim_{r \rightarrow \infty} \partial_\lambda (\sqrt{-g} V_{ab}^{\lambda\mu}) = 0$. Thus, we can get the total conservative angular momentum from (32) and (33)

$$J_{ab} = \int_{\Sigma_t} j_{ab}^\mu \sqrt{-g} d\Sigma_\mu = \frac{3c^3}{16\pi G} \int_{\partial \Sigma_t} \sqrt{-g} V_{ab}^{\mu\nu} d\sigma_{\mu\nu},$$

where $\sqrt{-g} d\Sigma_\mu$ is the covariant surface element of Σ_t , $d\Sigma_\mu = \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} dx^\nu \wedge dx^\lambda \wedge dx^\rho$, $d\sigma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} dx^\lambda \wedge dx^\rho$.

In summary, we have succeeded in obtaining an expression of an angular momentum conservation law in Riemann-Cartan space-time. This conservation law has the following main properties:

1. It is a covariant theory with respect to the generalized coordinate transformations, but the angular momentum tensor is not covariant under the local Lorentz transformation which, due to the equivalent principle, is reasonable to require.

2. For a closed system, the total angular momentum does not depend on the choice of the Riemannian coordinates and, according (34), the space-time at spatial infinity is flat, thus the conservative angular momentum J_{ab} should be a covariant object when we make a Lorentz transformation $\Lambda_{ab} = A_{ab} = \text{const.}$ at spatial infinity, as in special relativity

$$J'_{ab} = A_a^c A_b^d J_{cd}.$$

To understand this, the key point is that to obtain J_{ab} , one has to enclose everything of the closed system, and every point of space-time at spatial infinity belongs to the same Minkowski space-time in that region. This means that in general relativity for a closed system, the total angular momentum J_{ab} must be looked upon as a Lorentz tensor like that in special relativity.

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